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Canonical quantization theory of general singular QED system of Fermi field interaction with generally decomposed gauge potential

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HIGHLIGHTS

- We decompose the general gauge potential into two orthogonal parts according to general field theory.
- We identify a new approach for quantizing the general singular QED system.
- The results obtained are superior to those for the Lorentz gauge condition.
- The theory presented solves dilemmas such as the nucleon spin crisis.

Outline

1. Introduction

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3. Gauge-invariant decomposition of conjugate momenta

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mutators are permanently gauge-invariant. A system can only be measured in physical experiments when it is gauge-invariant. The vanishing longitudinal vector potential means that the gauge invariance of the general QED system cannot be retained. This is similar to the nucleon spin crisis dilemma, which is an example of a physical quantity that cannot be exactly measured experimentally. However, the theory here solves this dilemma by keeping the gauge invariance of the general QED system.

1. Introduction

Constraint theories about constrained Hamiltonian systems play a fundamental role in modern physics theory, especially field theories. A so-called constrained Hamiltonian system is a dynamical system described by a singular Lagrangian subjective to some inherent constraints in phase space [1]. All gauge field theories, without exception, belong to this category of constraint theories, such as QED, QCD, super gravity, super symmetry, and superstring theories, which dominate in particle physics. Thus, it is of considerable interest to investigate constraint theories. For a constraint system with a singular Lagrangian, standard quantization methods cannot be used [1,2] because it is nearly impossible to achieve quantization by solving the constraint equations to separate the independent variables. Dirac was the first to systematically investigate a constraint system with a singular Lagrangian [3,4]. Since then, many practical quantization programs have been developed and constraint theories for singular systems are still the focus of much research.

Gauge invariance is considered a first principle in modern physics. The form of any reasonable mathematical expression describing a real physical system should be invariant under an arbitrary gauge transformation. Experimentally, physical quantities can be measured only when they are gauge-invariant.

General gauge potential decomposition has been used to investigate issues related to the angular momentum of nucleons (i.e., the nucleon spin crisis) and similar problems in QED, and remarkable results have been achieved [5–9]. In these theories, the longitudinal part of a general gauge potential $\vec{A}$ does not vanish and contributes to the angular momenta of photon–electron interactions or even nucleons.

There are many studies of canonical quantization theories against different backgrounds [10–12]. However, quantization of a singular system requires suitable gauge conditions, of which the traditional Coulomb gauge is one. As a transverse gauge condition, the Coulomb gauge requires that the vector potential $\vec{A}$ is a spatially transverse field while the longitudinal part of $\vec{A}$ vanishes, that is, $\vec{A}_{\parallel} = 0$ [13,14]. However, if only the transverse part of the vector potential is used to describe the electromagnetic field interacting with a charged source, the observable electric field $\vec{E}$ is not guaranteed by the gauge invariance. This indicates that the longitudinal part of the vector potential cannot be considered to be zero in interaction field theory. This type of violation of the gauge invariance for a general QED system is similar to the nucleon spin crisis, whereby violation of the gauge invariance leads to inaccurate measurement of the nuclear angular momentum [9]. Thus, our aim was to search for a substitute for the traditional Coulomb gauge to facilitate quantization and retain the gauge invariance of the system.

Here we investigate the inherent constraints and canonical quantization theory for the general singular QED system of Fermi field interaction with a generally decomposed gauge potential, and solve the dilemma and restore the gauge invariance of the general QED system. The remainder of the paper is organized as follows. Section 2 describes the gauge-invariant Lagrangian density of the general QED system with a generally decomposed gauge potential. Section 3 investigates the violation of gauge invariance for the general QED system for the traditional Coulomb gauge condition. Section 4 presents the canonical Hamiltonian of the general QED system and inherent constraints. In Section 5, we find a suitable gauge condition to give the canonical quantization theory for the general singular QED system. All our results are consistent with current theories. The final section summarizes and concludes.

2. Gauge-invariant Lagrangian density of the general QED system with a generally decomposed gauge potential

The Lagrangian density of a QED system takes the form[1]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_e\left(\mathrm{i}\gamma^{\mu}D^e_{\mu} - m\right)\psi_e, \tag{2.1}$$

where $D^e_{\mu} = \partial_{\mu} + ieA_{\mu}$ is the covariate derivative, $F^{\mu\nu}$ is the electromagnetic field-strength tensor, which can be expressed as

$$F^{\mu\nu} = -F^{\nu\mu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}, \tag{2.2}$$

and $A^{\mu} = \left(A^0, \vec{A}\right)$ is four-dimensional gauge potential. As a general vector, the three-dimensional gauge potential $\vec{A}$ can be split into its longitudinal part $\vec{A}_{\parallel}$ and transverse part $\vec{A}_{\perp}$. These two orthogonal parts can be written as

$$A^i_{\parallel} = L^i_jA^j, \tag{2.3a}$$

$$A^i_{\perp} = T^i_jA^j, \tag{2.3b}$$

where L^i_j and T^i_j are longitudinal and transverse projection operators, respectively [5,6,13]:

$$L^i_j = \partial^i\frac{1}{\Delta}\partial_j, \tag{2.4a}$$

$$T^i_j = \delta^i_j - \partial^i\frac{1}{\Delta}\partial_j, \tag{2.4b}$$

where $\Delta = \partial_k\partial^k = -\nabla^2$. We can easily verify that

$$L^i_jT^j_k = 0, \qquad L^i_jL^j_k = L^i_k, \qquad T^i_jT^j_k = T^i_k. \tag{2.5}$$

Combining Eqs. (2.3)–(2.5), we can prove that

$$\epsilon_{ijk}\partial^jA^k_{\parallel} = \epsilon_{ijk}\partial^j\partial^k\frac{1}{\Delta}\partial_lA^l = 0, \tag{2.6a}$$

$$\partial_iA^i_{\perp} = \partial_i\left(\delta^i_j - \partial^i\frac{1}{\Delta}\partial_j\right)A^j = \partial_jA^j - \partial_jA^j = 0, \tag{2.6b}$$

which naturally yield

$$\nabla\times\vec{A}_{\parallel} = 0, \tag{2.7a}$$

$$\nabla\,.\,\vec{A}_{\perp} = 0. \tag{2.7b}$$

[1] We adopt the natural unit $c = \hbar = 1$; the flat space–time metric $g_{\mu\nu}=\mathrm{diag}(1-1-1-1)$; and the γ matrices $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0\end{pmatrix}$, where σ^i $(i = 1, 2, 3)$ are Pauli matrices. The summing rules comply with Einstein's summation, and the Greek index ranges from 0 to 3.

Using the generally decomposed gauge potentials, the field-strength tensor $F^{\mu\nu}$ can been expanded as

$$F^{i0} = \partial^i A^0 - \partial^0 A^i_{\parallel} - m\,\partial^0 A^i_{\perp}, \tag{2.8a}$$

$$F^{ij} = \partial^i A^j_{\perp} - \partial^j A^i_{\perp}. \tag{2.8b}$$

Eq. (2.8b) reveals that F^{ij} is only related to the transverse potential $\vec{A}_{\perp}$. To show why, we use Eq. (2.7a) to deduce

$$\vec{A}_{\parallel} = \nabla f, \tag{2.9}$$

where f is an arbitrary scalar function. Thus,

$$\partial^i A^j_{\parallel} - \partial^j A^i_{\parallel} = \partial^i \partial^j f - \partial^j \partial^i f = 0, \tag{2.10}$$

which implies that $\vec{A}_{\parallel}$ in the expression for F^{ij} vanishes, leaving only $\vec{A}_{\perp}$. In fact, Eqs. (2.8a) and (2.8b) are the electric and magnetic field strength, respectively, that is,

$$\vec{E} = -\nabla A_0 - \frac{\partial \vec{A}_{\parallel}}{\partial t} - \frac{\partial \vec{A}_{\perp}}{\partial t}, \tag{2.11a}$$

$$\vec{B} = \nabla \times \vec{A}_{\perp}. \tag{2.11b}$$

From (2.11) we find that the electric field $\vec{E}$ can also be split into its two orthogonal parts:

$$E^i_{\parallel} = \partial^i A^0 - \partial^0 A^i_{\parallel}, \tag{2.12a}$$

$$E^i_{\perp} = -\partial^0 A^i_{\perp}. \tag{2.12b}$$

Eqs. (2.11b) and (2.12b) show that both the magnetic field $\vec{B}$ and the transverse electric field $\vec{E}_{\perp}$ originate from the transverse potential $\vec{A}_{\perp}$.

Equipped with this knowledge, we can rewrite the Lagrangian density in Eq. (2.1) as

$$\mathcal{L} = -\frac{1}{2}F_{i0}F^{i0} - \frac{1}{4}F_{ij}F^{ij} + \bar{\psi}_e\left[\mathrm{i}\gamma^0 D^e_0 + \mathrm{i}\gamma^i\left(\partial_i + \mathrm{i}e\left(A_{i\perp} + A_{i\parallel}\right)\right) - m\right]\psi_e. \tag{2.13}$$

To show that the Lagrangian density (2.11) is gauge-invariant, we take the following gauge transformations [2,5,6,13,15,16]:

$$A'_0 = A_0 + \partial_0 \varepsilon, \tag{2.14a}$$

$$A'_{i\perp} = A_{i\perp}, \tag{2.14b}$$

$$A'_{i\parallel} = A_{i\parallel} + \partial_i \varepsilon, \tag{2.14c}$$

$$\psi'_e = e^{-\mathrm{i}e\varepsilon}\psi_e, \tag{2.14d}$$

$$\bar{\psi}'_e = \bar{\psi}_e e^{\mathrm{i}e\varepsilon}, \tag{2.14e}$$

where ε is an arbitrary scalar function. It is apparent that $A_{i\perp}$ is gauge-invariant, so all the parameters related only to $\vec{A}_{\perp}$ are also gauge-invariant (such as $\vec{B}$ and $\vec{E}_{\perp}$); A_0 and $\vec{A}_{\parallel}$ may also play the role in determining the gauge degrees of freedom. Using Eqs. (2.14), we can verify that

$$F'_{i0} = \partial_i A'_0 - \partial_0 A'_{i\parallel} - \partial_0 A'_{i\perp} = F_{i0}, \tag{2.15a}$$

$$F'_{ij} = \partial_i A'_{j\perp} - \partial_j A'_{i\perp} = F_{ij}, \tag{2.15b}$$

$$\begin{aligned}
\bar{\psi}'_e\left[\mathrm{i}\gamma^0 D'^e_0 + \mathrm{i}\gamma^i\left(\partial_i + \mathrm{i}eA'_{i\perp} + \mathrm{i}eA'_{i\parallel}\right) - m\right]\psi'_e &= \bar{\psi}_e[\mathrm{i}\gamma^0\left(\partial_0 - \mathrm{i}e\partial_0\varepsilon\right.\\
&\quad \left. +\ \mathrm{i}eA_0 + \mathrm{i}e\partial_0\varepsilon\right)\psi_e + \mathrm{i}\gamma^i\left(\partial_i - \mathrm{i}e\partial_i\varepsilon + \mathrm{i}eA_{i\perp} + \mathrm{i}eA_{i\parallel} + \mathrm{i}e\partial_i\varepsilon\right) - m]\psi_e \\
&= \bar{\psi}_e\left[\mathrm{i}\gamma^0 D^e_0 + \mathrm{i}\gamma^i\left(\partial_i + \mathrm{i}eA_{i\perp} + \mathrm{i}eA_{i\parallel}\right) - m\right]\psi_e,
\end{aligned} \tag{2.15c}$$

which are gauge-invariant and indicate that the Lagrangian density (2.13) is also gauge-invariant. Eqs. (2.15a) and (2.15b) mean that the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are gauge-invariant. For gauge transformations of the longitudinal electric field (2.12a), we also have

$$E_{\parallel}^{i\prime} = \partial^i A^{0\prime} - \partial^0 A_{\parallel}^{i\prime} = \partial^i \left(A^0 + \partial^0 \varepsilon\right) - \partial^0 \left(A_{\parallel}^i + \partial^i \varepsilon\right) = \partial^i A^0 - \partial^0 A_{\parallel}^i = E_{\parallel}^i, \tag{2.16}$$

which is also gauge-invariant. However, if the vector potential $\vec{A}$ is only taken as a transverse vector, the gauge transformation $A_i' = A_{i\perp}' = A_{i\perp} + \partial_i\varepsilon$ will not be rational because $A_{i\perp}$ and $\partial_i\varepsilon$ are parameters with different directions. This means that the longitudinal potential $\vec{A}_{\parallel}$ cannot vanish if an electric charge source exists, otherwise the electric field $\vec{E}_{\parallel}$ will not be gauge-invariant (in the case of no source, $A^0 = 0$, gauge invariance naturally requires that $\vec{A}_{\parallel} = 0$, i.e., $\vec{A} = \vec{A}_{\perp}$). However, if an electric charge source exists, $\vec{A}_{\parallel} = 0$ will mean that the gauge invariance of the general QED system will be violated. Therefore, the general requirement for $\vec{A}_{\parallel} \neq 0$ here is consistent.

Now we prove that $\vec{A}_{\perp}$ and $\vec{A}_{\parallel}$ are orthogonal:

$$\begin{aligned}\int d^3x A_{i\parallel} A_{\perp}^i &= \int d^3x \left(\partial_i f\right) A_{\perp}^i = \int d^3x [\partial_i \left(f A_{\perp}^i\right) - f\, \partial_i A_{\perp}^i] \\ &= \int d^3x \partial_i \left(f A_{\perp}^i\right) = \oint_{\infty} f \vec{A}_{\perp} \cdot d\vec{S} = 0,\end{aligned} \tag{2.17}$$

where we have used (2.9) ($A_{i\parallel} = \partial_i f$); $\oint_{\infty}$ represents the integral over the boundary of the whole space and $d\vec{S}$ is the area element vector of the integral surface. In the last step, the integrand $f\vec{A}_{\perp}$ is considered to vanish on the infinitely distant boundary and the integral over the boundary of whole space is set to zero, as is usual in field theory [13,15]. Thus, in later discussions, elimination of all the surface terms or total differential terms is appropriate, since these terms vanish on integration. Eq. (2.17) implies that $A_{\parallel}^i$ and $A_{\perp}^i$ are orthogonal in the sense of the integral over the whole space, but not pointwise. Under the gauge transformations (2.14), $A_{i\perp}'$ is also perpendicular to $A_{i\parallel}'$ as long as f takes the following gauge transformation:

$$f' = f + \varepsilon, \tag{2.18}$$

which can guarantee

$$\begin{aligned}\int d^3x A_{i\parallel}' A_{\perp}'^i &= \int d^3x (A_{i\parallel} + \partial_i \varepsilon) A_{\perp}^i = \int d^3x (\partial_i f + \partial_i \varepsilon) A_{\perp}^i \\ &= \int d^3x \partial_i \left(f + \varepsilon\right) A_{\perp}^i = \int d^3x \left(\partial_i f'\right) A_{\perp}^i = 0.\end{aligned} \tag{2.19}$$

Mathematically, the discussions above are strict. Any general vector can be represented by two orthogonal parts, but of course the vector in physics should not be excluded. Physically, Eqs. (2.7a) and (2.7b) are nothing but the transverse and longitudinal field conditions, respectively. The gauge invariance of the observable quantity $\vec{E}_{\parallel}$ is closely related to $\vec{A}_{\parallel}$ in the general QED system, and the condition $\vec{A}_{\parallel} = 0$ violates the gauge invariance.

3. Gauge-invariant decomposition of conjugate momenta

In Section 2, we obtained a gauge-invariant Lagrangian density (2.13), a gauge-invariant longitudinal electric field (2.12a) with $\vec{A}_{\parallel} \neq 0$, and the gauge-invariant magnetic field (2.11b) and transverse electric field (2.12b). Now we derive these observable field quantities using Eq. (2.13) once more.

Using Eqs. (2.8a) and (2.8b), we can easily expand the Lagrangian density (2.13) as

$$\begin{aligned}L =\ & \partial^i A^0 \partial_0 A_{i\parallel} - \frac{1}{2}\partial_i A_0 \partial^i A^0 - \frac{1}{2}\partial_0 A_{i\parallel} \partial^0 A_{\parallel}^i \\ & - \frac{1}{2}\partial_0 A_{i\perp} \partial^0 A_{\perp}^i - \frac{1}{4} F_{ij} F^{ij} + \bar{\psi}_e \left[\mathrm{i}\gamma^0 D_0^e + \mathrm{i}\gamma^i \left(\partial_i + \mathrm{i}e \left(A_{i\perp} + A_{i\parallel}\right)\right) - m \right] \psi_e,\end{aligned} \tag{3.1}$$

where the term $\partial^i A^0 \partial_0 A_{i\perp}$ and the cross term $\partial_0 A_{i\parallel} \partial^0 A^i_\perp$ are omitted, because both can be changed to a total differential, that is, surface terms.

If we consider all the field quantities as different variables and use Eq. (3.1), we obtain the canonical momenta conjugate to A_0, $A_{i\parallel}$, $A_{i\perp}$, ψ_e, and $\bar{\psi}_e$ as follows:

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0, \tag{3.2a}$$

$$\pi^i_\parallel = \frac{\partial L}{\partial \dot{A}_{i\parallel}} = \partial^i A^0 - \partial^0 A^i_\parallel, \tag{3.2b}$$

$$\pi^i_\perp = \frac{\partial L}{\partial \dot{A}_{i\perp}} = -\partial^0 A^i_\perp, \tag{3.2c}$$

$$\pi_{\psi_e} = \frac{\partial_r L}{\partial \dot{\psi}_e} = \bar{\psi}_e \mathrm{i}\gamma^0, \tag{3.2d}$$

$$\pi_{\overline{\psi}_e} = \frac{\partial_r L}{\partial \dot{\overline{\psi}}_e} = 0, \tag{3.2e}$$

where ∂_r is the right differential operator. Comparing Eqs. (3.2b) and (3.2c) with (2.12a) and (2.12b), we know that the canonical momenta $\pi^i_\parallel$ and $\pi^i_\perp$ are nothing but the electric field strengths, that is,

$$\pi^i_\parallel = E^i_\parallel, \tag{3.3a}$$

$$\pi^i_\perp = E^i_\perp, \tag{3.3b}$$

which naturally satisfy

$$\nabla \times \vec{\pi}_\parallel = \nabla \times \vec{E}_\parallel = 0, \tag{3.4a}$$

$$\nabla \cdot \vec{\pi}_\perp = \nabla \cdot \vec{E}_\perp = 0. \tag{3.4b}$$

Similar to Eq. (2.17), we can easily prove that $\pi^i_\parallel (E^i_\parallel)$ and $\pi^i_\perp (E^i_\perp)$ are orthogonal, that is,

$$\int d^3x \pi_{i\parallel} \pi^i_\perp = \int d^3x E_{i\parallel} E^i_\perp = 0. \tag{3.4c}$$

Explicitly, using Eqs. (3.2b) and (3.2c) we know that

$$\pi^i = \pi^i_\parallel + \pi^i_\perp = \partial^i A^0 - \partial^0 \left(A^i_\parallel + A^i_\perp\right) = \partial^i A^0 - \partial^0 A^i, \tag{3.5}$$

which is the canonical momentum conjugate to A_i, consistent with the result in field theories [13,15,14]. This indicates that it is reasonable to take A_0, $A_{i\parallel}$, and $A_{i\perp}$ as different variables. Using Eq. (3.5), we can prove that

$$\pi^i_\parallel = L^i_j \pi^j, \tag{3.6a}$$

$$\pi^i_\perp = T^i_j \pi^j, \tag{3.6b}$$

which further indicate that the two projection operators introduced in (2.4) are correct.

Besides, using Eq. (2.7b) we can easily show that

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \left(\vec{A}_\parallel + \vec{A}_\perp\right) = \nabla \times \vec{A}_\perp. \tag{3.7}$$

This result agrees with Eq. (2.12b) and further confirms the origination of the magnetic field $\vec{B}$.

The results obtained above are consistent with results in classical field theories, which confirms their correctness. Using the expansion of the Lagrangian density, we obtain the decomposed canonical momenta $\pi^i_\parallel$ and $\pi^i_\perp$, that is, the decomposed electric field strengths $E^i_\parallel$ and $E^i_\perp$. The equivalence naturally gives the gauge invariance property of $E^i_\parallel$ and $E^i_\perp$ to the canonical momenta $\pi^i_\parallel$ and $\pi^i_\perp$, respectively. The canonical momenta π^0, π_{ψ_e}, and $\pi_{\overline{\psi}_e}$, which are also gauge-invariant, serve as constraints in Section 4. Furthermore, Eqs. (3.2b) and (3.3a) explicitly indicate that $\vec{A}_\parallel$ has a conjugate relationship with $\vec{E}_\parallel$, which may imply some important properties of $\vec{A}_\parallel$ and is also a reason why we quantize the general QED system in a naturally deduced gauge condition about $\vec{A}_\parallel$ in Section 5.

4. Canonical Hamiltonian of general QED system and inherent constraints

In Section 3, we obtained canonical momenta conjugate to the field quantities. Now we continue to investigate the canonical Hamiltonian and the inherent constraints in the phase space. The singularity of the general QED system indicates that there must be constraints in the phase space. The canonical momenta (3.2a), (3.2d) and (3.2e) directly give rise to the primary constraints [2,17,14,18], expressed as

$$\phi^0 = \pi^0 \approx 0, \tag{4.1a}$$

$$\phi^0_{\psi_e} = \pi_{\psi_e} - \bar{\psi}_e \mathrm{i}\gamma^0 \approx 0, \tag{4.1b}$$

$$\phi^0_{\overline{\psi}_e} = \pi_{\overline{\psi}_e} \approx 0, \tag{4.1c}$$

where we adopt the weak equal symbol "$\approx$" according to the convention in constraint theories [19,20,16].

According to the Legendre transformation, we can easily obtain the canonical Hamiltonian density

$$\begin{aligned}
\mathcal{H}_c &= \pi^0\partial_0 A_0 + \left(\pi^i_\parallel + \pi^i_\perp\right)\partial_0(A_{i\parallel} + A_{i\perp}) + \pi_{\psi_e}\partial_0\psi_e + \pi_{\overline{\psi}_e}\partial_0\overline{\psi}_e - L \\
&= \pi^i_\parallel\partial_0 A_{i\parallel} + \pi^i_\perp\partial_0 A_{i\perp} + \pi_{\psi_e}\partial_0\psi_e - L \\
&= \pi^i_\parallel\partial_i A_0 - \frac{1}{2}\pi^i_\parallel\pi_{i\parallel} - \frac{1}{2}\pi^i_\perp\pi_{i\perp} + eA_0\overline{\psi}_e\gamma^0\psi_e + \frac{1}{4}F_{ij}F^{ij} \\
&\quad - \bar{\psi}_e\left[\mathrm{i}\gamma^i\left(\partial_i + \mathrm{i}e\left(A_{i\perp} + A_{i\parallel}\right)\right) - m\right]\psi_e \\
&= -\frac{1}{2}\pi^i_\parallel\pi_{i\parallel} - \frac{1}{2}\pi^i_\perp\pi_{i\perp} - A_0\left(\partial_i\pi^i_\parallel - e\overline{\psi}_e\gamma^0\psi_e\right) + \frac{1}{4}F_{ij}F^{ij} \\
&\quad - \bar{\psi}_e\left[\mathrm{i}\gamma^i\left(\partial_i + \mathrm{i}e\left(A_{i\perp} + A_{i\parallel}\right)\right) - m\right]\psi_e,
\end{aligned} \tag{4.2a}$$

where we have omitted $\pi^i_\parallel\partial_0 A_{i\perp}$ and $\pi^i_\perp\partial_0 A_{i\parallel}$, which can be transformed to surface terms, and $\partial_i\left(\pi^i_\parallel A_0\right)$. Note that the term $\partial_i\pi^i_\parallel - e\overline{\psi}_e\gamma^0\psi_e = 0$ in Eq. (4.2a) is just the Gauss law, which is presented later as a secondary constraint condition and indicates that A_0 plays the role of a Lagrange multiplier, that is, we naturally deduce the Gauss law. The canonical Hamiltonian of the general QED system is

$$\begin{aligned}
H_c &= \int d^3x\mathcal{H}_c \\
&= \int d^3x\left\{-\frac{1}{2}\pi^i_\parallel\pi_{i\parallel} - \frac{1}{2}\pi^i_\perp\pi_{i\perp} - A_0\left(\partial_i\pi^i_\parallel - e\overline{\psi}_e\gamma^0\psi_e\right) + \frac{1}{4}F_{ij}F^{ij}\right. \\
&\quad \left. - \bar{\psi}_e\left[\mathrm{i}\gamma^i\left(\partial_i + \mathrm{i}e\left(A_{i\perp} + A_{i\parallel}\right)\right) - m\right]\psi_e\right\}.
\end{aligned} \tag{4.2b}$$

Inserting the primary constraints (4.1a)–(4.1c) into (4.2b), we obtain the total Hamiltonian

$$\begin{aligned}
H_T &= \int d^3x(\mathcal{H}_c + \lambda_1\phi^0 + \lambda_2\phi^0_{\psi_e} + \lambda_3\phi^0_{\overline{\psi}_e}) \\
&= \int d^3x\left\{-\frac{1}{2}\pi^i_\parallel\pi_{i\parallel} - \frac{1}{2}\pi^i_\perp\pi_{i\perp} - A_0\left(\partial_i\pi^i_\parallel - e\overline{\psi}_e\gamma^0\psi_e\right) + \frac{1}{4}F_{ij}F^{ij}\right. \\
&\quad \left. - \bar{\psi}_e\left[\mathrm{i}\gamma^i\left(\partial_i + \mathrm{i}e\left(A_{i\perp} + A_{i\parallel}\right)\right) - m\right]\psi_e + \lambda_1\phi^0 + \lambda_2\phi^0_{\psi_e} + \lambda_3\phi^0_{\overline{\psi}_e}\right\},
\end{aligned} \tag{4.2c}$$

where λ_i are Lagrange multipliers.

According to the Dirac–Bergmann algorithm [4,21] for obtaining the Hamiltonian formulation of systems with constraints, we should utilize the Poisson bracket [13,17,14], which can be defined as

$$\left\{F(x), G(x')\right\}_P = \int d^3x'' \left[\frac{\delta F(x)}{\delta\varphi(x'')}\frac{\delta G(x')}{\delta\pi_\varphi(x'')} - \frac{\delta G(x')}{\delta\varphi(x'')}\frac{\delta F(x)}{\delta\pi_\varphi(x'')}\right], \tag{4.3a}$$

which is appropriate for boson quantities. However, for fermion quantities, Eq. (4.3a) should be modified as [1]

$$\left\{F(x), G(x')\right\}_P = \int d^3x'' \left[\frac{\partial_r F(x)}{\partial\varphi(x'')}\frac{\partial_l G(x')}{\partial\pi_\varphi(x'')} - (-1)^{n_F n_G}\frac{\partial_r G(x')}{\partial\varphi(x'')}\frac{\partial_l F(x)}{\partial\pi_\varphi(x'')}\right], \tag{4.3b}$$

where ∂_l is the left differential operator, and n_F and n_G are Grassmann parities.

Using Eqs. (4.3a) and (4.3b), the Poisson brackets for all the nonvanishing canonical variables can be calculated as

$$\left\{A_0(x), \pi^0(x')\right\}_P = \delta^3(x-x'), \tag{4.4a}$$

$$\left\{A_{i\parallel}(x), \pi^j_\parallel(x')\right\}_P = L^j_i\delta^3(x-x'). \tag{4.4b}$$

Appendix A describes how (4.4b) is deduced.

$$\left\{A_{i\perp}(x), \pi^j_\perp(x')\right\}_P = T^j_i\delta^3(x-x'), \tag{4.4c}$$

$$\left\{\psi_e(x), \pi_{\psi_e}(x')\right\}_P = \delta^3(x-x'), \tag{4.4d}$$

$$\left\{\bar\psi_e(x), \pi_{\bar\psi_e}(x')\right\}_P = \delta^3(x-x'), \tag{4.4e}$$

where we must consider the dependence of variations between A_i and π^i and their orthogonal components.

We can also compute the consistency conditions for the primary constraint [2,20] using Eq. (4.4) and the Poisson bracket as follows[2]:

$$\begin{aligned}\dot\phi^0(x) &= \left\{\phi^0(x), H_T(x')\right\}_P \\ &= \partial_i\pi^i_\parallel(x) - e\bar\psi_e(x)\gamma^0\psi_e(x) \approx 0,\end{aligned} \tag{4.5a}$$

$$\begin{aligned}\dot\phi^0_{\psi_e}(x) &= \left\{\phi^0_{\psi_e}(x), H_T(x')\right\}_P \\ &= A_0(x)e\bar\psi_e(x)\gamma^0 + e\bar\psi_e(x)\gamma^i\left(A_{i\parallel}(x) + A_{i\perp}(x)\right) + \mathrm{i}\partial_i\left(\bar\psi_e(x)\gamma^i\right) \\ &\quad - m\bar\psi_e(x) + \mathrm{i}\gamma^0\lambda_3 \approx 0.\end{aligned} \tag{4.5b}$$

Appendix B describes the calculation of Eq. (4.5b). Similarly, we have

$$\begin{aligned}\dot\phi^0_{\bar\psi_e}(x) &= \left\{\phi^0_{\bar\psi_e}(x), H_T(x')\right\}_P \\ &= A_0(x)e\gamma^0\psi_e(x) - \mathrm{i}\gamma^i\partial_i\psi_e(x) + e\gamma^i\left(A_{i\parallel}(x) + A_{i\perp}(x)\right)\psi_e(x) \\ &\quad + m\bar\psi_e(x) + \mathrm{i}\gamma^0\lambda_2 \approx 0.\end{aligned} \tag{4.5c}$$

To ensure that the primary constraint (4.5a) is preserved over time, Eq. (4.5a) must be imposed as the secondary constraint, that is,

$$\phi^1 = \partial_i\pi^i_\parallel - e\bar\psi_e\gamma^0\psi_e \approx 0. \tag{4.6}$$

[2] To distinguish differentials with respect to x and x', we use the notation $\partial_i = \frac{\partial}{\partial x^i}$ and $\partial^{x'}_i = \frac{\partial}{\partial x'^i}$ in subsequent discussions.

We note that Eqs. (4.5b) and (4.5c) determine the Lagrange multipliers λ_2 and λ_3, and they do not produce new constraints. Using the condition $\partial_i \pi_{\parallel}^i = \partial_i \left(E^i - E_{\perp}^i\right) = \partial_i E^i$, we know that Eq. (4.6) is just the Gauss law, as mentioned above, with charge density

$$\rho_e = -e\bar{\psi}_e \gamma^0 \psi_e. \tag{4.7}$$

According to Dirac's theory, all the constraints should be separated into first-class and second-class constraints. If the Poisson brackets of a constraint and all other constraints vanish, then this constraint is called a first-class constraint, otherwise it is called a second-class constraint. Hence, we calculate the Poisson brackets for all the constraints as follows:

$$\left\{\phi^0(x), \phi^0_{\psi_e}(x')\right\}_P = \left\{\phi^0(x), \phi^0_{\bar{\psi}_e}(x')\right\}_P = \left\{\phi^0(x), \phi^1(x')\right\}_P = 0, \tag{4.8a}$$

$$\left\{\phi^0_{\psi_e}(x), \phi^0_{\bar{\psi}_e}(x')\right\}_P = \left\{\pi_{\psi_e}(x) - \bar{\psi}_e(x)\,\mathrm{i}\gamma^0, \pi_{\bar{\psi}_e}(x')\right\}_P = -\mathrm{i}\gamma^0\delta^3(x-x'), \tag{4.8b}$$

$$\begin{aligned}\left\{\phi^0_{\psi_e}(x), \phi^1(x')\right\}_P &= \left\{\pi_{\psi_e}(x) - \bar{\psi}_e(x)\,\mathrm{i}\gamma^0, \partial_i^{x'}\pi_{\parallel}^i(x') - e\bar{\psi}_e(x')\gamma^0\psi_e(x')\right\}_P \\ &= -e\bar{\psi}_e(x)\gamma^0\delta^3(x-x'),\end{aligned} \tag{4.8c}$$

$$\begin{aligned}\left\{\phi^0_{\bar{\psi}_e}(x), \phi^1(x')\right\}_P &= \left\{\pi_{\bar{\psi}_e}(x), \partial_i^{x'}\pi_{\parallel}^i(x') - e\bar{\psi}_e(x')\gamma^0\psi_e(x')\right\}_P \\ &= e\gamma^0\psi_e(x)\,\delta^3\left(x-x'\right),\end{aligned} \tag{4.8d}$$

which show that ϕ^0 is of first class, whereas $\phi^0_{\psi_e}$, $\phi^0_{\bar{\psi}_e}$, and ϕ^1 are of second class. However, ϕ^1 is a first-class constraint for a free electromagnetic field, which means that ϕ^1 should be changed to a first-class constraint via the linear combination

$$\Lambda_2 = \phi^1 + \mathrm{ie}\left(\phi^0_{\psi_e}\psi_e + \bar{\psi}_e\phi^0_{\bar{\psi}_e}\right) = \partial_i\pi_{\parallel}^i + \mathrm{ie}\left(\pi_{\psi_e}\psi_e + \bar{\psi}_e\pi_{\bar{\psi}_e}\right). \tag{4.9}$$

Using the two equations

$$\left\{\phi^0_{\psi_e}(x), \Lambda_2(x')\right\}_P = \mathrm{ie}\phi^0_{\psi_e}(x)\,\delta^3\left(x-x'\right) \approx 0, \tag{4.10a}$$

$$\left\{\phi^0_{\bar{\psi}_e}(x), \Lambda_2(x')\right\}_P = -\mathrm{ie}\phi^0_{\bar{\psi}_e}(x)\,\delta^3\left(x-x'\right) \approx 0, \tag{4.10b}$$

we can show that Λ_2 is a first-class constraint.

Now we have deduced all the inherent constraints, which consist of two first-class constraints $\left(\Lambda_1 = \phi^0, \Lambda_2\right)$ and two second-class constraints $(\phi^0_{\psi_e}, \phi^0_{\bar{\psi}_e})$. From the above discussion, we know that Poisson brackets are crucial to the derivation of the two types of constraint. Moreover, from the investigations in Section 3 and Eqs. (4.1) and (4.6), it is easy to verify that the Hamiltonians (4.2b) and (4.2c) are also gauge-invariant under the condition $\vec{A}_{\parallel} \neq 0$.

5. Canonical quantization of the general singular QED system

In Section 4, we deduced that the general QED system has two first-class and two second-class constraints. The existence of these constraints means that standard quantization methods can no longer be used [1,2]. Therefore, we use the Dirac quantization procedure [2,19,22] to achieve quantization of the general singular QED system.

To quantize the system we should choose some suitable gauge conditions. The existence of two first-class constraints in the phase space means that two gauge conditions [2] are required to remove the gauge degrees of freedom [1]. To keep the gauge invariance of the system, we cannot use the traditional Coulomb gauge in which the longitudinal vector potential vanishes. Hence, similar to the choice of a gauge condition in field theories, a gauge condition can be chosen as

$$\Omega_2 = \partial_i A_{\parallel}^i \approx 0. \tag{5.1}$$

For an arbitrary longitudinal field $\vec{A}_\parallel$, even if it does not satisfy (5.1), we always can choose a condition $\partial_i\partial^i\varepsilon = 0$ or $\partial_i\partial^i\varepsilon = -\partial_i A^i_\parallel$ so that the transformed longitudinal field $A'^i_\parallel$ meets the relation $\partial_i A'^i_\parallel = \partial_i\left(A^i_\parallel + \partial^i\varepsilon\right) = 0$. Using Eq. (2.9), we also can rewrite Eq. (5.1) as $\Omega_2 = \partial_i\partial^i f \approx 0$, which is just a Laplace equation, where f ought to be a harmonic function. Hence, combining Eqs. (2.7a) and (5.1), we can show that the longitudinal potential $\vec{A}_\parallel$ is actually a harmonic field, whereas Eq. (5.1) can be called a simplified Coulomb gauge. The consistency condition for Ω_2 is

$$\begin{aligned}\dot{\Omega}_2(x) &= \partial_i\dot{A}^i_\parallel(x) = \left\{\partial_i A^i_\parallel(x), H_T(x')\right\}_P \\ &= -\partial_i\pi^i_\parallel(x) + \partial_i\partial^i A^0(x) \approx 0,\end{aligned} \tag{5.2}$$

where we have ignored the surface term $\int d^3x'\partial^{x'}_j\left(A_0(x')\,\pi^j_\parallel(x')\right)$. Eq. (5.2) serves as another gauge condition[3]:

$$\Omega'_1 = \partial_i\pi^i_\parallel - \partial_i\partial^i A_0 \approx 0, \tag{5.3a}$$

which can be represented as

$$\Omega_1 = A^0 - \frac{\partial_i\pi^i_\parallel}{\Delta} \approx 0. \tag{5.3b}$$

Using Eqs. (4.6) and (4.7), (5.3a) can be rewritten as

$$\partial_i\partial^i A_0 = \rho_e, \tag{5.4a}$$

which just is the Poisson equation and can be solved by

$$A^0(x) = \frac{1}{4\pi}\int d^3x'\,\frac{\rho_e(x')}{|\vec{x}-\vec{x}'|}. \tag{5.4b}$$

Now we have obtained all the constraints and gauge conditions as Eqs. (4.1a)–(4.1c), (4.9), (5.1) and (5.3), and we represent them as $\Phi_i = (\Omega_1, \Lambda_1, \Omega_2, \Lambda_2, \phi^0_{\psi_e}, \phi^0_{\bar{\psi}_e})$. Using these constraints and gauge conditions, we can obtain the matrix

$$\begin{aligned}C(x,x') &= \left\{\Phi_i(x), \Phi_j(x')\right\}_P \\ &= \begin{pmatrix} 0 & 1 & \dfrac{\partial^{x'}_i\partial^i}{\Delta} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{\partial^{x'}_i\partial^i}{\Delta} & 0 & 0 & \partial^{x'}_i\partial^i & 0 & 0 \\ 0 & 0 & -\partial^{x'}_i\partial^i & 0 & -ie\phi^0_{\psi_e}(x) & ie\phi^0_{\bar{\psi}_e}(x) \\ 0 & 0 & 0 & ie\phi^0_{\psi_e}(x) & 0 & -i\gamma^0 \\ 0 & 0 & 0 & -ie\phi^0_{\bar{\psi}_e}(x) & i\gamma^0 & 0 \end{pmatrix}\delta^3(x-x').\end{aligned} \tag{5.5}$$

Obviously, Eq. (5.5) is invertible and indicates that all the gauge conditions and constraints are of second class [2,19,22]. The Lagrange multiplier λ_1 in the total Hamiltonian (4.2c) can be determined as

$$\begin{aligned}\left\{\Omega_1(x), H_T(x')\right\}_p &= \left\{A^0(x) - \frac{\partial_i}{\Delta}\pi^i_\parallel(x), \int d^3x'\left(e\bar{\psi}_e\gamma^j A_{j\parallel}\psi_e + \lambda_1\pi^0\right)(x')\right\}_P \\ &= \lambda_1 + \frac{\partial_i}{\Delta}\left(e\bar{\psi}_e\gamma^i\psi_e\right)(x) = \lambda_1 - \frac{1}{\Delta}\nabla\cdot\vec{j}(x) \approx 0.\end{aligned} \tag{5.6}$$

[3] In fact, $\partial_i\pi^i_\parallel - \partial_i\partial^i A_0 = \partial_i\left(\partial^i A^0 - \partial^0 A^i_\parallel\right) - \partial_i\partial^i A_0 = \partial^0\partial_i A^i_\parallel \approx 0$, that is, $\partial_i A^i_\parallel \approx 0$.

The inverse matrix of Eq. (5.5) is

$$
C^{-1}\left(x,x'\right) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{\Delta} & \frac{e\phi^0_{\bar\psi_e}(x)\lambda^0}{\Delta} & \frac{e\phi^0_{\psi_e}(x)\lambda^0}{\Delta} \\ 0 & 0 & 0 & -\frac{1}{\partial^{x'}_i\partial^i} & -\frac{e\phi^0_{\bar\psi_e}(x)\gamma^0}{\partial^{x'}_i\partial^i} & -\frac{e\phi^0_{\psi_e}(x)\gamma^0}{\partial^{x'}_i\partial^i} \\ 0 & -\frac{1}{\Delta} & \frac{1}{\partial^{x'}_i\partial^i} & 0 & 0 & 0 \\ 0 & -\frac{e\phi^0_{\bar\psi_e}(x)\gamma^0}{\Delta} & \frac{e\phi^0_{\bar\psi_e}(x)\gamma^0}{\partial^{x'}_i\partial^i} & 0 & 0 & \mathrm{i}\gamma^0 \\ 0 & -\frac{e\phi^0_{\psi_e}(x)\gamma^0}{\Delta} & \frac{e\phi^0_{\psi_e}(x)\gamma^0}{\partial^{x'}_i\partial^i} & 0 & \mathrm{i}\gamma^0 & 0 \end{pmatrix} \times\delta^3\left(x-x'\right). \tag{5.7}
$$

To achieve quantization, the Poisson bracket should be replaced by a Dirac bracket [1,2,13,17,14,22], which, using Eq. (5.7), can be defined as

$$
\begin{aligned}
\left\{F(x),G\left(x'\right)\right\}_D &= \left\{F(x),G(x')\right\}_P \\
&\quad - \int d^3y\int d^3y'\,\{F(x),\Phi_i(y)\}_P\,C^{-1}_{ij}\left(y,y'\right)\left\{\Phi_j\left(y'\right),G\left(x'\right)\right\}_P .
\end{aligned} \tag{5.8}
$$

Now we arrive at the fundamental Dirac brackets of the general QED system, with respect to which the general QED system can be quantized:

$$
\begin{aligned}
\left\{A_{i\perp}(x),\pi^j_\perp\left(x'\right)\right\}_D &= \left\{A_{i\perp}(x),\pi^j_\perp\left(x'\right)\right\}_P \\
&\quad - \int d^3y\int d^3y'\,\{A_{i\perp}(x),\Phi_k(y)\}_P\,C^{-1}_{km}\left(y,y'\right)\left\{\Phi_m\left(y'\right),\pi^j_\perp\left(x'\right)\right\}_P \\
&= \left\{A_{i\perp}(x),\pi^j_\perp\left(x'\right)\right\}_P = T^j_i\delta^3\left(x-x'\right),
\end{aligned} \tag{5.9a}
$$

$$
\left\{A_{i\parallel}(x),\pi^j_\parallel\left(x'\right)\right\}_D = 0. \tag{5.9b}
$$

Appendix C shows how to deduce Eq. (5.9b).

$$
\begin{aligned}
&\left\{\psi_e(x),\bar\psi_e\left(x'\right)\right\}_D \\
&= \left\{\psi_e(x),\bar\psi_e\left(x'\right)\right\}_P - \int d^3y\int d^3y'\,\{\psi_e(x),\Phi_k(y)\}_P\,C^{-1}_{km}\left(y,y'\right)\left\{\Phi_m\left(y'\right),\bar\psi_e\left(x'\right)\right\}_P \\
&= -\int d^3y\int d^3y'\left(\{\psi_e(x),\Phi_k(y)\}_P\,C^{-1}_{km}\left(y,y'\right)\left\{\Phi_m\left(y'\right),\bar\psi_e\left(x'\right)\right\}_P\right)_{\substack{k=5\\m=6}} \\
&= -\int d^3y\left\{\psi_e(x),\pi_{\psi_e}(y)\right\}_P \mathrm{i}\gamma^0\left\{\pi_{\bar\psi_e}(y),\bar\psi_e\left(x'\right)\right\}_P \\
&= -\mathrm{i}\gamma^0\delta^3\left(x-x'\right),
\end{aligned} \tag{5.9c}
$$

$$
\begin{aligned}
&\left\{\psi_e(x),\pi_{\psi_e}\left(x'\right)\right\}_D \\
&= \left\{\psi_e(x),\pi_{\psi_e}\left(x'\right)\right\}_P - \int d^3y\int d^3y'\,\{\psi_e(x),\Phi_k(y)\}_P\,C^{-1}_{km}\left(y,y'\right)\left\{\Phi_m\left(y'\right),\pi_{\psi_e}\left(x'\right)\right\}_P \\
&= \left\{\psi_e(x),\pi_{\psi_e}\left(x'\right)\right\}_P = \delta^3\left(x-x'\right).
\end{aligned} \tag{5.9d}
$$

All the other unwritten Dirac brackets vanish. Using Eq. (3.2d), we can easily see that Eq. (5.9d) is equivalent to Eq. (5.9c), which reflects the consistency of our theory. The right-hand side of Eq. (5.9a) just is the transverse delta function [2,13]

$$\delta^{3\,j}_{\perp i}\left(x-x'\right)=T^j_i\delta^3\left(x-x'\right). \tag{5.10}$$

From Eqs. (5.9), we find that all the Dirac brackets related to the longitudinal vector potential $\vec{A}_{\parallel}$ are zero. The results above are identical to those for the traditional Coulomb gauge, because the quantization in different gauge conditions is equivalent [2,22], but the traditional Coulomb gauge is incompatible with the longitudinal vector potential $\vec{A}_{\parallel}$. Our simplified Coulomb gauge condition (5.1) is developed from the traditional Coulomb gauge and it fixes the longitudinal vector potential $\vec{A}_{\parallel}$, which keeps the gauge invariance of the general QED system.

Using the relation $i\{F,G\}_D\rightarrow FG-(-1)^{n_F n_G}GF$ [1], we can easily obtain the independent nonvanishing quantum commutators as

$$\left[\hat{A}_{i\perp}(x),\hat{\pi}^j_{\perp}\left(x'\right)\right]_-=\mathrm{i}T^j_i\delta^3\left(x-x'\right), \tag{5.11a}$$

$$\left[\hat{\psi}_e(x),\hat{\pi}_{\psi_e}\left(x'\right)\right]_+=\mathrm{i}\delta^3\left(x-x'\right), \tag{5.11b}$$

where $[.,.]_-$ denotes a commutator, $[.,.]_+$ denotes an anticommutator, and the Grassmann parity of a boson is zero. Comparison of these commutators with those in general field theories reveals that all the non-zero quantum commutators are gauge-invariant, which is very important for investigating gauge-invariant systems.

6. Summary and conclusion

1.We present a new approach to quantize the general singular QED system by decomposing a general gauge potential into two orthogonal components in general field theory.
2.Using these two orthogonal components, we obtain an expansion of the gauge-invariant Lagrangian density, from which we deduce the two orthogonal canonical momenta conjugate to the two components of the gauge potential. We then obtain the canonical Hamiltonian in the phase space and deduce the inherent constraints.
3.In terms of the naturally deduced gauge condition, the quantization results are exactly consistent with those in the traditional Coulomb gauge condition and superior to those in the Lorentz gauge condition.
4. We find that all the nonvanishing quantum commutators are permanently gauge-invariant. A system can only be measured in physical experiments when it is gauge-invariant.
5.The vanishing longitudinal vector potential means that the gauge invariance of the general QED system cannot be retained. This is similar to the nucleon spin crisis dilemma, which is an example of a physical quantity that cannot be exactly measured experimentally. However, the theory here solves this dilemma by keeping the gauge invariance of the general QED system.

canonical quantization of such a constrained system. It not only guarantees the gauge invariance of the system but also yields very explicit and simple canonical quantization results, and can be applied to other systems to achieve canonical quantization. By bringing in two projection operators [5,6,13], we successfully divide the general gauge potential into two orthogonal components that meet the conditions $\nabla \times \vec{A}_{\parallel} = 0$ and $\nabla \cdot \vec{A}_{\perp} = 0$. Using the two orthogonal components, we expand the Lagrangian density, from which we deduce an orthogonal decomposition of the canonical momentum. The transverse component $\vec{A}_{\perp}$ only contributes to the transverse part of the electric field strength, that is, $\vec{E}_{\perp} = -\partial^0 \vec{A}_{\perp}$, which is consistent with results for the traditional Coulomb gauge. The longitudinal component $\vec{A}_{\parallel}$ only contributes to the longitudinal part of the electric field strength, that is, $\vec{E}_{\parallel} = \nabla A^0 - \partial^0 \vec{A}_{\parallel}$, which means that $\vec{E}_{\parallel}$ has a canonical conjugate relation with $\vec{A}_{\parallel}$.

From the canonical conjugate momentum, we obtain the primary constraints (ϕ^0, $\phi^0_{\psi_e}$, $\phi^0_{\bar{\psi}_e}$) and the canonical Hamiltonian H_c in phase space, which can be combined into a total Hamiltonian H_T. The consistence condition $\left\{\phi^0, H_T\right\}_p \approx 0$ serves as the secondary constraint condition ϕ^1, which is just the Gauss law. The consistence conditions for $\phi^0_{\psi_e}$ and $\phi^0_{\bar{\psi}_e}$ can determine the Lagrange multipliers but not lead to new constraints. The primary constraint ϕ^0 is a first-class constraint, while $\phi^0_{\psi_e}$, $\phi^0_{\bar{\psi}_e}$, and ϕ^1 are second-class constraints, but ϕ^1 should be changed to a first-class constraint via the linear combination $\Lambda_2 = \phi^1 + \mathrm{i}e\left(\phi^0_{\psi_e}\psi_e + \bar{\psi}_e\phi^0_{\bar{\psi}_e}\right)$. Thus, $\Lambda_1 = \phi^0$ and Λ_2 constitute first-class constraints, while $\phi^0_{\psi_e}$ and $\phi^0_{\bar{\psi}_e}$ are second-class constraints.

The Lagrangian density, the Hamiltonian, and the two orthogonal components of the electric field strength and the magnetic field strength retain the gauge invariance property, which still holds even if we take the gauge condition $\Omega_2 = \partial_i A^i_{\parallel} \approx 0$. Moreover, all the non-zero quantum commutators are also gauge-invariant.

The existence of constraints means that quantization of the system is quite difficult. Because of the two first-class constraints, we must choose two gauge conditions [2]. Here we choose a naturally deduced gauge condition $\Omega_2 = \partial_i A^i_{\parallel} \approx 0$, whose consistence condition gives rise to the gauge condition $\Omega_1 = A^0 - \frac{\partial_i}{\Delta}\pi^i_{\parallel} \approx 0$. Thus, using Dirac's quantization procedure, we achieve an expectational canonical quantization. The nonvanishing commutator for bosons reserves the transverse parts $\left[A_{i\perp}(x), \pi^j_{\perp}(x')\right]_- = T^j_i \delta^3\left(x - x'\right)$, which is equivalent to results for the traditional Coulomb gauge [13]. The identical results can be attributed to the gauge condition $\Omega_2 = \partial_i A^i_{\parallel} \approx 0$, which, unlike the traditional Coulomb gauge, does not require the condition $A^i_{\parallel} = 0$. Thus, all the physically observable quantities of the general QED system interacting with electric charges retain gauge invariance. Although the Lorentz gauge condition can also guarantee gauge invariance for the system, nonphysical parameters are required to impose constraints on physical states. By contrast, our quantization theory does not suffer from this problem and can directly lead to ideal results.

A nuclear system that cannot retain gauge invariance leads to the nucleon spin crisis, whereby real physical quantities cannot be exactly measured experimentally [5–9]. A similar dilemma caused by the vanishing longitudinal vector potential $\vec{A}_{\parallel}$ in general QED systems or other systems means that gauge invariance cannot be guaranteed for some exactly measurable quantities in physical experiments. By using the decomposed gauge potential and the gauge condition (5.1), we retain the gauge invariance of the physical quantities of the general QED system, such as the electric field strength.

Our results for general QED systems are very simple and useful and can be applied to different quantum systems such as condensed physics, atomic physics, molecular physics, quantum optics, quantum field theory, particle physics, nuclear physics, and strong laser interactions with matter. Investigations and applications in these branches of science have been described in the literature [25,26]. In ongoing research we are generalizing the results for general QED systems to general QCD systems.

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Appendix A

$$
\begin{aligned}
&\left\{A_{i\parallel}(x), \pi_{\parallel}^{j}\left(x'\right)\right\}_P = \int d^3x'' \left[\frac{\delta A_{i\parallel}(x)}{\delta A_l(x'')} \frac{\delta A_l\left(x''\right)}{\delta A_{k\parallel}(x'')} \frac{\delta \pi_{\parallel}^{j}\left(x'\right)}{\delta \pi^m(x'')} \frac{\delta \pi^m\left(x''\right)}{\delta \pi_{\parallel}^{k}(x'')}\right] \\
&= \int d^3x'' \left[\frac{g_{ik'}L_{k''}^{k'}g^{k''n}\delta A_n(x)}{\delta A_l(x'')} \frac{\delta\left(A_{l\parallel}+A_{l\perp}\right)\left(x''\right)}{\delta A_{k\parallel}(x'')} \frac{L_{j'}^{j}\delta\pi^{j'}\left(x'\right)}{\delta \pi^m(x'')} \frac{\delta\left(\pi_{\perp}^{m}+\pi_{\parallel}^{m}\right)\left(x''\right)}{\delta \pi_{\parallel}^{k}(x'')}\right] \\
&= \int d^3x'' g_{ik'}g^{k''n}L_{k''}^{k'}L_{j'}^{j}\delta_n^l\delta_l^k\delta_m^{j'}\delta_k^m\delta^3\left(x-x''\right)\delta^3\left(x''-x'\right) \\
&= g_{ik'}g^{k''n}L_{k''}^{k'}L_n^j\delta^3(x-x') \\
&= \partial_i\frac{1}{\Delta}\partial_{k''}\partial^j\frac{1}{\Delta}\partial^{k''}\delta^3(x-x') = L_i^j\delta^3\left(x-x'\right).
\end{aligned}
$$

Appendix B

$$
\begin{aligned}
\dot{\phi}_{\psi_e}^0 &= \left\{\phi_{\psi_e}^0(x), H_T\left(x'\right)\right\}_P \\
&= \left\{\pi_{\psi_e}(x) - \bar{\psi}_e(x)\,\mathrm{i}\gamma^0, H_T\left(x'\right)\right\}_P \\
&= \Bigg\{\pi_{\psi_e}(x), \int d^3x' \Big[A_0\left(x'\right)e\overline{\psi}_e\left(x'\right)\gamma^0\psi_e\left(x'\right) - \overline{\psi}_e\left(x'\right)\gamma^i\partial_i^{x'}\psi_e\left(x'\right) \\
&\quad + m\bar{\psi}_e\left(x'\right)\psi_e\left(x'\right) + e\bar{\psi}_e\left(x'\right)\gamma^i\left(A_{i\perp}\left(x'\right)+A_{i\parallel}\left(x'\right)\right)\psi_e\left(x'\right)\Big]\Bigg\}_P \\
&\quad - \left\{\bar{\psi}_e(x)\,\mathrm{i}\gamma^0, \int d^3x'\lambda_3\pi_{\bar{\psi}_e}\left(x'\right)\right\}_P \\
&= -\int d^3x''\int d^3x' \Bigg\{\Big[A_0\left(x'\right)e\overline{\psi}_e\left(x'\right)\gamma^0 + e\bar{\psi}_e\left(x'\right)\left(A_{i\perp}\left(x'\right)+A_{i\parallel}\left(x'\right)\right) \\
&\quad + \mathrm{i}\partial_i^{x'}\left(\overline{\psi}_e\left(x'\right)\gamma^i\right) - m\bar{\psi}_e\left(x'\right)\Big](-1)^{n_{\pi_{\psi_e}}n_{\psi_e}}\frac{\partial_r\psi_e\left(x'\right)}{\partial\psi_e(x'')}\frac{\partial_l\pi_{\psi_e}(x)}{\partial\pi_{\psi_e}(x'')} \\
&\quad + \mathrm{i}\gamma^0\lambda_3\frac{\partial_r\bar{\psi}_e(x)}{\partial\bar{\psi}_e(x'')}\frac{\partial_l\pi_{\bar{\psi}_e}\left(x'\right)}{\partial\pi_{\overline{\psi}_e}(x'')}\Bigg\} \\
&= \int d^3x''\int d^3x'\delta^3\left(x-x''\right)\delta^3\left(x'-x''\right)\Big[A_0\left(x'\right)\; e\overline{\psi}_e\left(x'\right)\gamma^0 \\
&\quad + e\bar{\psi}_e\left(x'\right)\left(A_{i\perp}\left(x'\right)+A_{i\parallel}\left(x'\right)\right) - m\bar{\psi}_e\left(x'\right) + \mathrm{i}\partial_i^{x'}\left(\overline{\psi}_e\left(x'\right)\gamma^i\right) + \mathrm{i}\gamma^0\lambda_3\Big] \\
&= A_0(x)\,e\bar{\psi}_e(x)\gamma^0 + e\bar{\psi}_e(x)\gamma^i\left(A_{i\parallel}(x)+A_{i\perp}(x)\right) \\
&\quad + \mathrm{i}\partial_i\left(\bar{\psi}_e(x)\gamma^i\right) - m\bar{\psi}_e(x) + \mathrm{i}\gamma^0\lambda_3 \approx 0.
\end{aligned}
$$

Appendix C

$$
\begin{aligned}
&\left\{A_{i\parallel}(x), \pi_{\parallel}^{j}(x')\right\}_{D} \\
&= \left\{A_{i\parallel}(x), \pi_{\parallel}^{j}(x')\right\}_{P} - \int d^3y \int d^3y' \left\{A_{i\parallel}(x), \Phi_k(y)\right\}_P C_{km}^{-1}(y, y') \left\{\Phi_m(y'), \pi_{\parallel}^{j}(x')\right\}_P \\
&= L_i^j \delta^3(x - x') - \int d^3y \int d^3y' \left(\left\{A_{i\parallel}(x), \Phi_k(y)\right\}_P C_{km}^{-1}(y, y') \left\{\Phi_m(y'), \pi_{\parallel}^{j}(x')\right\}_P\right)_{\substack{k=1,4\\ m=3}} \\
&= L_i^j \delta^3(x - x') - \int d^3y \int d^3y' \left\{A_{i\parallel}(x), \Lambda_2(y)\right\}_P C_{43}^{-1}(y, y') \left\{\Omega_2(y'), \pi_{\parallel}^{j}(x')\right\}_P \\
&= L_i^j \delta^3(x - x') - \int d^3y \int d^3y' \left\{A_{i\parallel}(x), \partial_k^y \pi_{\parallel}^{k}(y)\right\}_P \frac{\delta^3(y - y')}{\partial_{ny'}\partial^{ny}} \left\{\partial_m^{y'} A_{\parallel}^{m}(y'), \pi_{\parallel}^{j}(x')\right\}_P \\
&= L_i^j \delta^3(x - x') - \int d^3y \partial_k^y L_i^k \delta^3(x - y) \frac{1}{\partial_{ny}\partial^{ny}} \partial_m^y g^{ml} L_l^j \delta^3(y - x') \\
&= L_i^j \delta^3(x - x') - \partial_k L_i^k \frac{1}{\Delta} \partial^l L_l^j \delta^3(x - x') \\
&= L_i^j \delta^3(x - x') - \partial_k \partial^k \frac{1}{\Delta} \partial_i \frac{1}{\Delta} \partial^l \partial^j \frac{1}{\Delta} \partial_l \delta^3(x - x') \\
&= L_i^j \delta^3(x - x') - L_i^j \delta^3(x - x') = 0.
\end{aligned}
$$

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